Tangent Spaces of Differentiable Manifolds

M C^∞ differentiable manifold, $p \in M$, consider set of all C^∞ curves γ such that $\gamma : (a, b) \to M$ $0 \in (a, b)$, define equivalence relation $\gamma_1 \sim \gamma_2$ $\Longleftrightarrow$ γ_1's Taylor expansion in one and hence any coordinate system around p up to but not including 2nd order terms = same thing for γ_2.

Detail: $(x_1, \ldots x_n)$ coord. sys defined in a nbhd of p, $p \leftrightarrow (x_1^0, \ldots x_n^0)$ then

$$\gamma(t) = (x_1^0, \ldots, x_n^0) + t\left(\frac{dx_1(\gamma(t))}{dt}, \ldots, \frac{dx_n(\gamma(t))}{dt}\right)$$

+ higher than 1st order.

So T. expansion up to but not including 2nd order involves only p and $\frac{dx_1(\gamma(t))}{dt}, \ldots, \frac{dx_n(\gamma(t))}{dt}$.

But in another coordinate system $(\hat{x}_1, \ldots, \hat{x}_n)$ which has $\hat{x}_i = \hat{x}_i(x_1, \ldots, x_n)$ (notation) we have

$$\frac{d}{dt}\hat{x}_j(\gamma(t)) = \sum \frac{\partial \hat{x}_j}{\partial x_i} \cdot \frac{dx_i(\gamma(t))}{dt}$$

by Chain Rule so knowing first order part in x coordinates determines first order part in $\hat{x}$ coordinates.

Definition: Tangent space at p, notation T_pM (sometimes M_p) $=$ (set of all such γ) / $\sim$ equivalence relation

Additive structure: Choose coordinates around p with $p \leftrightarrow (0, \ldots, 0)$. Define $[\gamma_1] + [\gamma_2]$ $= [\gamma_1 + \gamma_2]$ where this + denotes vector addition in coordinates.

Check: Same as adding first order parts of Taylor expansion, hence (since transformation from one coordinate system to another is linear) independent of coordinate choice.

Similarly for $\alpha[\gamma]$.

Tangent space is vector space.

Given $(x_1, \dots, x_n)$ coordinates define

$$\frac{\partial}{\partial x_i}\Big|_p = \text{tangent vector of } \gamma(t) = (x_1^0, \dots, x_i^0 + t, \dots x_n)$$

Then

$\frac{\partial}{\partial x_1}\Big|_p, \dots \frac{\partial}{\partial x_n}\Big|_p$ is a vector space basis for T_pM. (Exercise) Use (for independence)

Operation on funcs: $[\gamma] f = \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$, $\gamma(0) = p$

~~Transformation~~ So $[\gamma] x_i$ = a_i-component of 1st order Taylor expansion

$\left(\sum a_j \frac{\partial}{\partial x_j}\right) x_i = a_i$ so linear comb $= 0 \Rightarrow$ coefficients $= 0$.

Transformation Rule (important)

$$\frac{\partial}{\partial \hat{x}_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial \hat{x}_i} \frac{\partial}{\partial x_j}$$

Proof: Works on functions by chain rule. Follows, both sides have same components

Example: Polar coord $x = r\cos\theta$ $y = r\sin\theta$

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

(visualize and interpret geometrically!)

Exercise (1) $\frac{\partial}{\partial x} = ?\,\frac{\partial}{\partial r} + ??\,\frac{\partial}{\partial \theta}$

Same for $\frac{\partial}{\partial \theta} =$ Interpret geometrically again

(2) Put second set of formulas back into first set to verify.

Lie bracket and vector fields:

Vector field = function V from M into $\bigcup_{p\in M} T_pM$ such that $V(p) \in T_pM$. (p varies)

In local coordinates

$$V(p) = \sum_{j=1}^{n} f_j(p) \frac{\partial}{\partial x_j}\Big|_p$$

So have idea of vector field being continuous or C^∞ or C^k k finite.

Note that $\bigcup_{p\in M} T_pM$ is itself a manifold in a natural way: $\bigcup_{p \in \text{domain of } (x_1, \dots x_n)} T_pM$ has coordinates $(\underbrace{x_1, \dots x_n}_{\text{for } p}, \underbrace{a_1 \dots a_n}_{\text{for } v\in T_pM})$

$v = \sum a_j \frac{\partial}{\partial x_j}$

M C^∞ $\Rightarrow$ chart overlap C^∞ (linear transformation on a's, coefficients depending on $(x_1, \dots, x_n)$)

Exercise (1) Write the transformation out explicitly.

(2) $a_1(x,y)\frac{\partial}{\partial x} + a_2(x,y)\frac{\partial}{\partial y}$ on open set in $\mathbb{R}^2$ is what in $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ terms?

Lie bracket: If X, Y are C^∞ vector fields defined in a nbhd of $p \in M$ then $\exists$ a unique vector $v \in T_pM$ such that for all C^∞ functions F : nbhd of $p \to \mathbb{R}$

$$vF = X(YF)\big|_p - Y(Xf)\big|_p .$$

Proof: Write $X = \sum a_i \frac{\partial}{\partial x_i}$ $\quad Y = \sum b_j \frac{\partial}{\partial x_j}$

$(x_1, \dots, x_n)$ coordinates around p, a, b, C^∞ functions then

$$X(YF) = \sum_{i,j} a_i \frac{\partial}{\partial x_i}\left(b_j \frac{\partial F}{\partial x_j}\right)$$

$$= \sum a_i b_j \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum a_i \frac{\partial b_j}{\partial x_i}\frac{\partial F}{\partial x_j}$$

$$Y(XF) = \sum a_j b_i \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum b_i \frac{\partial a_j}{\partial x_i}\frac{\partial F}{\partial x_j}$$

So (interchanging i, j in this sum and $\frac{\partial^2 F}{\partial\,\partial}$ sums

$$X(Yf) - Y(Xf) = \sum_j \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial F}{\partial x_j}$$

So

$$v = \sum_j \left(\sum_i \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\right)\Big|_p \frac{\partial}{\partial x_j}\Big|_p$$

works. $\square$ (Uniqueness is clear: use $F = x_j$

Note: Existence of v used coordinates but defining formula $vF = X(YF)$ has no coords so v is ind of coord choice

Definition: X, Y vector fields. then

$[X,Y]$ = vector field with value at p = v constructed as just done.

Lemma: X, Y $C^\infty \Rightarrow [X,Y)$ C^∞
Proof: Look at formula in coordinates.

$[X,Y]$ is additively linear and constant linear in each slot.
Function behavior: $[fX, Y] = f[X,Y] - (Yf)X$

$$[X, gY] = g[X,Y] + (Xg)Y.$$

Check: $[fX, Y]F = (fX)(YF) - Y((fX)F)$

$$= f(X(YF)) - (fX)YF - (Yf)(XF)$$
$$= f[X,Y] - (Yf)X.$$

These formulas enable one to figure out Lie brackets easily (usually), e.g. on $\mathbb{R}^2$

$$\left.\begin{aligned}\left[f\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] &= 0 - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\\ \left[\frac{\partial}{\partial x}, g\frac{\partial}{\partial y}\right] &= 0 + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}\end{aligned}\right\} \text{ since } \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0.$$

Later: $V_1, \dots V_n$ vector fields, linearly ind at each pt in nbhd of p with all Lie brackets $\equiv 0$ on nbhd $\Rightarrow$ on possibly smaller nbhd $\exists$ $x_1, \dots x_n$ coordinates $\ni$ $V_i = \frac{\partial}{\partial x_i}$ all i in nbhd.